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RPN 16

SC-RR-68-536

EXACT SOLUTION OF POISSON'S EQUATION FOR ONE-DIMENSIONAL, SPACE-CHARGE-LIMITED, RELATIVISTIC FLOW

Jack E. Boers
Dermod Kelleher
Radiation Source Research Division 9143
Sandia Laboratories, Albuquerque

November 1968

ABSTRACT

The problem of space-charge-limited flow in a relativistic planar diode has been solved exactly. The approach used is to solve Poisson's equation for space-charge-limited flow, assuming the initial velocities of the accelerated particles are zero, through the use of two power series convergent in the voltage range $0 \leq V \leq 1.022$ MV and $1.022 \text{ MV} \leq V < \infty$.

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EXACT SOLUTION OF POISSON'S EQUATION FOR ONE-DIMENSIONAL, SPACE-CHARGE-LIMITED, RELATIVISTIC FLOW

Introduction

Space-charge-limited flow in relativistic planar diodes has been the subject of study by many authors.[1, 2, 3, 4] However, their solutions are either not exact over a full range of diode voltages or are not convenient to use over a wide voltage range. In this paper, Poisson's equation, assuming that the initial velocities of the accelerated particles are zero, is solved through the use of two power series convergent in the voltage range $0 \le V \le 1.022$ MV and $1.022 \text{ MV} \le V < \infty$. The approach is straightforward, and the resulting series solutions are easy to use and rapidly convergent. Rationalized MKS units are used throughout.

Solution Development

The space-charge-limited flow in a planar diode is governed by Poisson's equation, written in one dimension:

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_o} , \tag{1}$$

were ρ is the space-charge density, ϵ_o the permittivity of free space, and V the potential.

The current density, J, is given by $J = -\rho v$, where v is the particle velocity. It is convenient to work with a normalized voltage U:

$$U = \frac{V}{V_o} = \frac{ZeV}{m_o c^2} , \tag{2}$$

where Ze is the electronic charge, m_o the particle rest mass, and c the velocity of light.

Rewriting Poisson's equation in terms of U and using the relativistic relation between voltage and velocity,

$$v = c \frac{(U^2 + 2U)^{1/2}}{U + 1} ,$$

Eq. (1) becomes

$$\frac{d^2U}{dx^2} = \frac{J}{V_o \epsilon_o c} \frac{U + 1}{(U^2 + 2U)^{1/2}} \quad . \tag{3}$$

Integrating once and applying the boundary condition that $\frac{dU}{dx} = 0$ at $x = 0$, one obtains

$$\frac{dU}{dx} = \xi(U^2 + 2U)^{1/4} \quad ,$$

or

$$\frac{dx}{dU} = \frac{1}{\xi} (U^2 + 2U)^{-1/4} \quad , \tag{4}$$

where $\xi = \left\{\frac{2J}{V_o \epsilon_o c}\right\}^{1/2}$. It is at this point that most authors have introduced approximations, either by putting restrictions on the size of U(e.g., $U << 1$ or $U >> 1$), or by forming approximate series solutions which are extremely complicated and have a limited range of applicability. Acton's solution,[4] because of its slow convergence, is not particularly useful above 1 MV.

Using the substitution $\frac{2x}{\xi} = (U^2 + 2U)^{-1/4} y$, Eq. (4) becomes

$$2(U^2 + 2U) \frac{dy}{dU} - y(U + 1) = (U^2 + 2U). \tag{5}$$

The solution to the homogeneous form of Eq. (5) is obtained by direct integration, and is $x_c = \frac{C_1}{\xi}$. The solution to the nonhomogeneous form is obtained by expanding y in a power series in U,

$$y = \sum_{o}^{\infty} a_n U^n,$$

and proceeding in the usual manner. The results obtained are

$$\begin{aligned} a_o &= 0 \\ a_1 &= 2/3 \\ a_2 &= -1/21 \\ &\vdots \\ a_n &= -a_{n-1} \frac{2n - 3}{4n - 1}, \quad n \geq 3 \quad . \end{aligned} \tag{6}$$

The series is seen to be absolutely convergent for $U < 2.0$, conditionally convergent at $U = 2.0$, and divergent for $U > 2.0$. The complete solution to Eq. (5) is thus written:

$$x = \frac{1}{\xi}\left\{\frac{2}{(U^2 + 2U)^{1/4}} \sum_{o}^{\infty} a_n U^n + C_1\right\} . \tag{7}$$

Since $U = 0$ at $x = 0$, this implies that C_1 is identically zero.

The solution can be cast in a more convenient form

$$Jx^2\,(\text{Amps}) = \frac{2m_o c^3\, \epsilon_o}{Ze(U^2 + 2U)^{1/2}}\left\{\frac{2}{3}\,U - \frac{U^2}{21} + \cdots\right.$$

$$\left. + (-1)^{n+1}\, \frac{1\cdot 3\cdot 5 \cdots (2n-5)(2n-3)}{21\cdot 11\cdot 15 \cdots (4n-5)(4n-1)}\, U^n \cdots\right\}^2 . \tag{8}$$

Equation (8) is the exact solution for the relativistic planar diode in the region $0 \leq U \leq 2.0$; eg., $0 \leq V \leq 1.022$ MV. Notice that as $U \to 0$,

$Jx^2 \approx \frac{m_o c^3\, \epsilon_o}{Ze} \frac{8}{9\sqrt{2}} U^{3/2}$, which is the familiar Child result for low voltage charged particle flow.

To arrive at a solution for $U > 2.0$, Eq. (5) can be transformed by letting $U = \frac{1}{R}$, becoming

$$2(2R^3 + R^2)\,\frac{dy}{dR} + (R^2 + R)y = -(2R + 1) . \tag{9}$$

The homogeneous form of Eq. (9) is also solved by direct integration, giving the solution $x_c = \frac{C_2}{\xi}$. The solution to the nonhomogeneous equation is obtained by expanding y in a power series in R of the form

$$y = R^s \sum_{o}^{\infty} b_m R^m$$

and proceeding as before. The results are

$$
\begin{aligned}
s &= -1 \\
b_o &= 1 \\
b_1 &= 1 \\
&\vdots \\
b_m &= -b_{m-1}\left(\frac{4m-7}{2m-1}\right), \quad m \geq 2 \quad .
\end{aligned}
\tag{10}
$$

The series is seen to be absolutely convergent for $U > 2.0$, conditionally convergent at $U = 2.0$, and divergent for $U < 2.0$.

The complete solution to Eq. (9) is written, in terms of U, as

$$
x = \frac{1}{\xi}\left\{\frac{2}{(U^2 + 2U)^{1/4}} \sum_{m=o}^{\infty} b_m U^{-m+1} + C_2\right\} \quad . \tag{11}
$$

The constant C_2 equals -0.847213 (this determination is given in the next section). Again, it is convenient to put Eq. (11) in the same form as Eq. (8):

$$
Jx^2\,(\text{Amps}) = \frac{2m_o c^3 \epsilon_o}{Ze}\left\{\frac{1}{(U^2+2U)^{1/4}}\left[U + 1 - \frac{1}{3U} + \cdots + (-1)^m \frac{1\cdot5\cdot9\cdots(4m-1)(4m-7)}{3\cdot5\cdot7\cdots(2m-3)(2m-1)}\, U^{-m+1}\right] - 0.847213\right\}^2 \quad . \tag{12}
$$

Equation (12) is the exact solution for the relativistic planar diode in the region $2.0 \leq U < \infty$; e.g., $1.022\ \text{MV} \leq V < \infty$.

As is seen, as $U \to \infty$,

$$
Jx^2 \approx \frac{2m_o c^3 \epsilon_o}{Ze}\, U, \tag{13}
$$

which is the ultrarelativistic solution that can be obtained from Eq. (4) by direct integration if $(U^2 + 2U)^{1/4}$ is approximated by $U^{1/2}$.

Evaluation of the Series

The constant, C_2, in Eq. (11) was determined by equating the two solutions at U = 2.0. The series are slowly convergent at this point, and the constant was determined as a function of the number of terms employed in the two series. The constant changes by a factor of 10^{-5} between 1000 and 10,000 terms and is asymptotically approaching $-0.847213 \pm 1 \times 10^{-6}$. The evaluations below all employ this value for the constant.

As shown in Table I, convergence of the series is quite rapid except near U = 2.0. Only four terms of the series are required to be within 1 percent of the exact value for U greater than 3.0 or less than 1.3. Two terms are adequate for U greater than 10.0; e.g., 5 MV for electrons.

TABLE I

Number of Terms Required for Jx^2 to be Within 1 Percent of Exact Value for Different Values of U

U	Number of Terms
0.1	2
0.5	2
1.0	3
1.5	5
1.8	8
2.0	13
	---- from Eq. (8)
2.0	12
2.2	7
2.5	5
3.0	4
4.0	3
10.0	2
20.0	2
50.0	2
100.0	2
200.0	1
	---- from Eq. (12)

The rapid convergence of the series is further shown in Figure 1; here, the value of the series is plotted versus the number of terms employed. For U = 4.0, the series has converged (for plotting purposes) after four terms of the series have been computed. Even at U = 2.0 the error is less than 5 percent for four terms. The convergence for the lower series is very similar.

The percent error obtained by employing only the first two terms is shown in Figure 2. Here, the error is less than 1 percent for U greater than 10.0 or less than 0.7. Additional terms uniformly decrease the error. The error can be reduced to less than 1 percent at any point by employing 13 terms of the series. This will reduce the error above 9.0 and below 0.5 to less than 10^{-9} percent.

In Figure 3 the exact solution, as well as the low and high potential approximations, is plotted. At U = 10.0, both approximations are more than 60 percent in error. Child's law is within 1 percent up to a few kilovolts, and the ultrarelativistic solution is within 1 percent only for U greater than 200.0, e.g., 100 MV for electrons. The error in using only the first two terms of the series is significantly less than that obtained by employing either of the limiting expressions. A table of values for these solutions is presented in the Appendix.

Summary

Poisson's equation has been solved exactly for the case of a relativistic, space-charge-limited planar diode. The solution is presented in the form of two rapidly converging series which reduce to the classical low and high potential expressions in the limits. Expressions employing only the first two terms of the power series substantially reduce, over a wide range of potential, the errors which result from using either of the limiting expressions. Much greater accuracy can be obtained by employing more terms of the series.

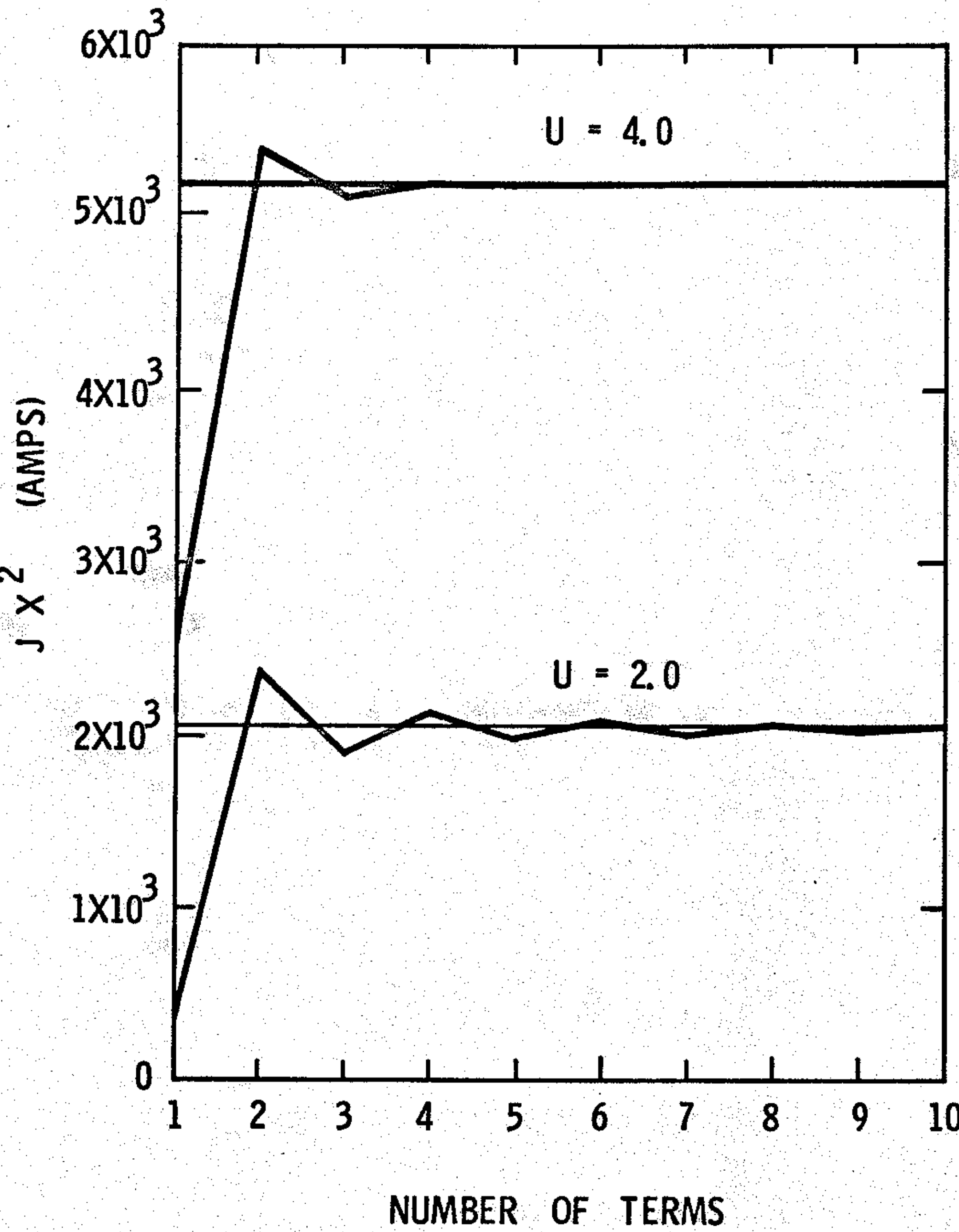

Figure 1. Evaluation of Jx^2 for the first 10 terms of the series for U = 2.0 and 4.0 with the exact values given by the horizontal lines

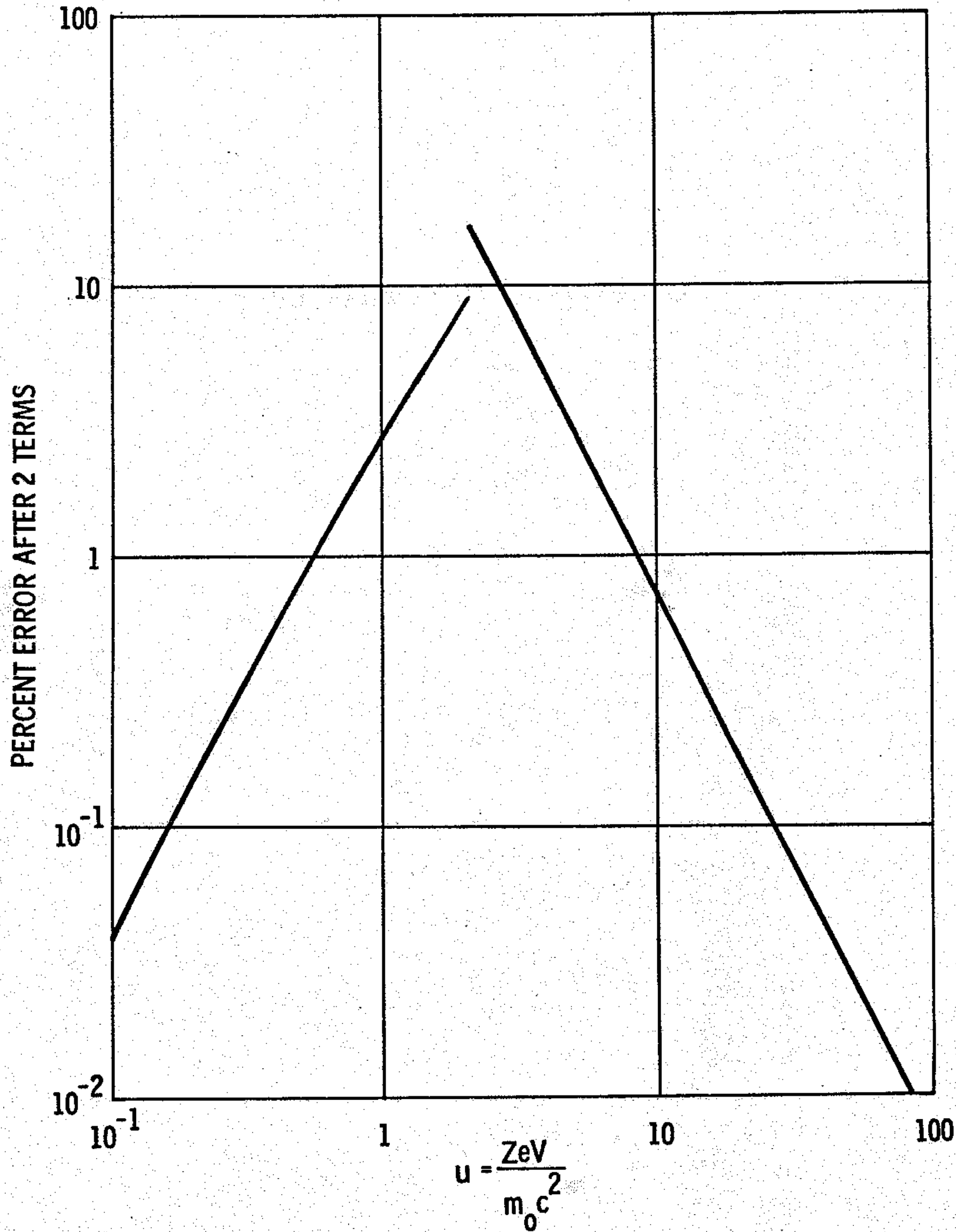

Figure 2. Error introduced by keeping only first two terms in exact power series solutions

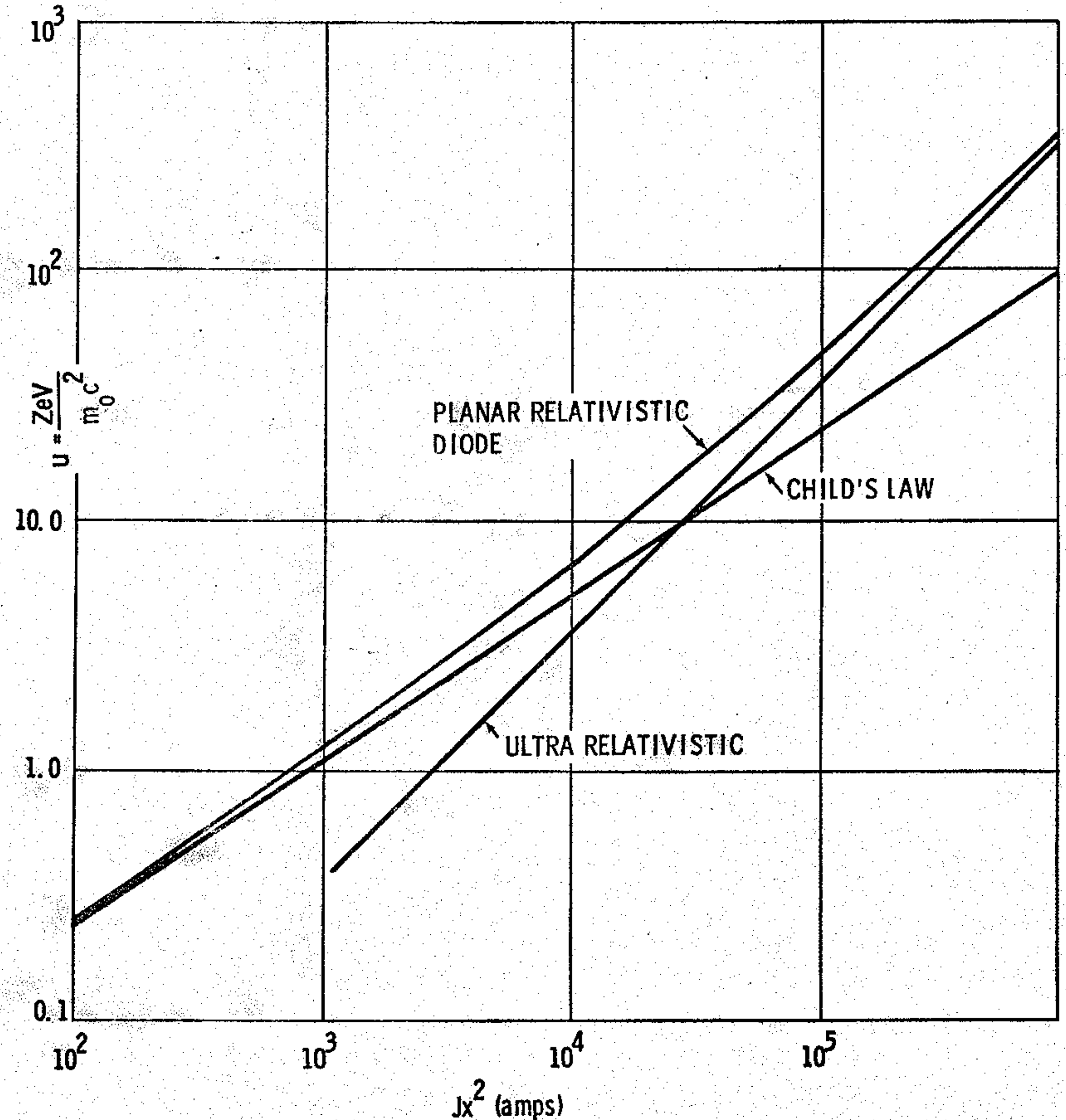

Figure 3. Comparison of planar relativistic diode, Child's law, and ultrarelativistic solutions

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APPENDIX

Representative Results

Comparisons of the values for Jx^2 using the exact solution, Child's law, and the ultrarelativistic solution, are tabulated below for U ranging from 0.1 to 98.0. For the exact solution, 100 terms in the series were used, averaging the last 2 terms. Equation (8) was used to obtain Jx^2 for $U \leq 2.0$, and Eq. (12) for $U \geq 2.0$. Solutions of both equations for $U = 2.0$ are presented for comparison purposes.

J*XSQ

U	EXACT	CHILD'S	ULTRARELATIVISTIC
0.1	2.672974449+001	2.701281984+001	2.718112564+002
0.2	7.483644009+001	7.640379235+001	5.436225128+002
0.3	1.361323246+002	1.403627292+002	8.154337692+002
0.4	2.075913165+002	2.161025587+002	1.087245026+003
0.5	2.874313888+002	3.020125071+002	1.359056282+003
0.6	3.744369266+002	3.970057507+002	1.630867538+003
0.7	4.677106485+002	5.002844245+002	1.902678795+003
0.8	5.665564380+002	6.112303388+002	2.174490051+003
0.9	6.704153135+002	7.293461356+002	2.446301308+003
1.0	7.788268934+002	8.542203671+002	2.718112564+003
1.1	8.914045465+002	9.855052672+002	2.989923820+003
1.2	1.007818514+003	1.122901834+003	3.261735077+003
1.3	1.127783972+003	1.266149391+003	3.533546333+003
1.4	1.251052305+003	1.415018036+003	3.805357590+003
1.5	1.377404561+003	1.569303021+003	4.077168846+003
1.6	1.506646427+003	1.728820470+003	4.348980102+003
1.7	1.638604303+003	1.893404060+003	4.620791359+003
1.8	1.773122174+003	2.062902393+003	4.892602615+003
1.9	1.910058822+003	2.237176885+003	5.164413872+003
2.0	2.049277804+003	2.416100057+003	5.436225128+003
2.0	2.049280699+003	2.416100057+003	5.436225128+003
6.0	8.748206838+003	1.255442416+004	1.630867538+004
10.0	1.654431838+004	2.701281984+004	2.718112564+004
14.0	2.486148112+004	4.474679926+004	3.805357590+004
18.0	3.350037570+004	6.523470154+004	4.892602615+004
22.0	4.236346826+004	8.814627079+004	5.979847641+004
26.0	5.139436095+004	1.132478443+005	7.067092666+004
30.0	6.055695449+004	1.403627293+005	8.154337692+004
34.0	6.982648997+004	1.693512075+005	9.241582718+004
38.0	7.918511150+004	2.000991837+005	1.032882774+005
42.0	8.861943918+004	2.325111894+005	1.141607277+005
46.0	9.811914023+004	2.665058028+005	1.250331779+005
50.0	1.076760378+005	3.020125071+005	1.359056282+005
54.0	1.172835292+005	3.389694525+005	1.467780785+005
58.0	1.269361920+005	3.773218128+005	1.576505287+005
62.0	1.366295082+005	4.170205496+005	1.685229790+005
66.0	1.463596661+005	4.580214585+005	1.793954292+005
70.0	1.561234146+005	5.002844244+005	1.902678795+005
74.0	1.659179544+005	5.437728271+005	2.011403297+005
78.0	1.757408548+005	5.884530607+005	2.120127800+005
82.0	1.855899887+005	6.342941421+005	2.228854302+005
86.0	1.954634824+005	6.812673863+005	2.337576805+005
90.0	2.053596752+005	7.293461355+005	2.446301308+005
94.0	2.152770869+005	7.785055311+005	2.555025810+005
98.0	2.252143914+005	8.287223194+005	2.663750313+005

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